FROM SUPERGRAVITY TO BALL BEARINGS. COHOMOLOGY FOR PHYSICISTS AND ENGINEERS

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ABSTRACT. A *Mathematica*-based package SUPERLie for the study of Lie (super)algebras and their cohomology is offered. Among applications we find: (1) calculation of the Riemannian tensors needed to write Supergravity Equations for any N-extended Minkowski superspace and to classify possible models for these superspaces; (2) the possibility to study stability of nonholonomic systems (ballbearings, gyroscopes, electro-mechanical devices like rotor collector with a gliding contact; differential games or pursuit problems; waves in plasma, etc.); (3) description of the analogue of the curvature tensor for nonlinear nonholonomic constraints and the fields of solids or their surfaces, e.g., cones, as in optimal control; (4) a new method for the study of integrability of differential equations (numerical methods can provide with individual solutions but are unable, generally, to study qualitative behavior, e.g., stability); (5) lists of "natural" operators, i.e., the operators between sections of tensor fields (or jets) invariant with respect to the group of diffeomorphism or its subgroup.

For nonholonomic systems the formula for the analog of curvature tensor (the sign of whose components determines stability) is new.

SUPERLIE makes it possible to determine (1) Lie algebras via defining relations, from Cartan matrix, realize via vector fields, as polynomials with Poisson or contact (Legendre) bracket, etc., (2) various modules over these Lie algebras (tensors, with vacuum vector, etc.), (3) list central extensions and deformations.

All the above problems can be expressed in terms of Lie algebra cohomology.

Disclaimer. The authors are not experts in either analytical mechanics or physics. Therefore our examples are very simple and scanty (Internet lists thousands of entries for nonholonomic). We were just lucky to observe that various problems listed below (we tried to provide with ample references to start with) can be solved (at least theoretically) by means of computing certain Lie algebra cohomology. The talk will require no preliminary knowledge and will be addressed mainly to students who do not know too much to be prejudiced against unusual methods with only few romantic results to back up our claims of their usefulness.

Here is the story (fairy tale).

§1. NONHOLONOMIC DYNAMICS: HISTORY

Supergravity. In 1972 Leites introduced [L1] the what is now called *superscheme* or *algebraic supervariety*. (Unfortunately, short communications to Russian mathematical Surveys were not translated into English at that time, but, even if they were, the text was too short. Even its expansion rewritten from algebraic language to C^∞ one [BL] did not get much acclaim.) This (and [L2]) was an answer to a question of Berezin together with whom he was slightly ahead of the time, see Minlos' recollections in [D].

Now supermanifold theory is fully recognized thanks to its part, *supersymmetry*, considered in theoretical physics as the language of the future unified theory of all fundamental forces. Wess and Zumino were the first to realize and explain to others that **we live on a supermanifold** and thus made one of the most fundamental steps in science, not only

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in physics. Physicists realized that there is not one, but a whole hierarchy of *standard* Minkowski *super*spaces and a number exsotic models (see [WB], [SS], [Ma]), labeled for the standard model by an inner parameter N running (conventionally) 1 to 8, or ad infinitum in recent dissident studies of M. Vasiliev [Va]. The most interesting was supposed to be the largest model, for $N = 8$.

Ensuing enthusiasm was overwhelming (at least, among physicists). Classification of simple finite dimensional Lie superalgebras performed by Kaplansky, Nahm–Rittenberg–Scheunert and skillfully rounded up by Kac convinced many mathematicians that this is "it" in a sense.

Mass media (from Pravda to New York Times) heralded that soon Einstein's dream — Grand Unified Theory (now called SUSY GUTs for its supersymmetry) — will come true. But eventually fanfare subsided. The reason: SUGRA equations for $N = 1$ were established by 1976, $N = 2$ (and to an extent $N = 3$) were supposed to be written in 1984. After much work in various research centers, it was Ogievetsky's group who realized that to fulfill natural requirements (see §4) one has to *enlarge* Minkowski space underlying Minkowski superspace. How to do it for $N > 3$ was unclear.

To apply for grants subsidizing studies of nonexisting equations became more difficult and most of the researchers left the field. What was the problem?

One of the best experts, Wess, honestly lectured [WB]: we do not know how to express the Riemann tensor (the left hand side of equations to be written) in the general case.

In [Wi], Witten wrote: "*Direct experimental confirmation of supersymmetry is one of the prime missions of the proposed Superconducting supercollider.*" And then added: "*More fundamentally, I believe that the main obstacle is that the core geometrical ideas — which must underline string theory the way Riemannian geometry underlines general relativity — have not yet been unearthed.*"

We believe that Leites was lucky to have discovered some of these ideas. They are related to (a) *nonholonomic* mechanics, (b) the presentation of the curvature tensor via cohomology and (c) the classification of simple Lie superalgebras of vector fields — the analog of É. Cartan's classification, [LS].

Relation with nonholonomic mechanics is occasioned by the fact that Minkowski *super*spaces (whatever it is, see [WB], [SS]; there is a hierarchy of models, labeled by an inner parameter N running (conventionally) 1 to 8, or ad infinitum in recent dissident studies [Va]) are **nonholonomic**. The importance of Lie algebra cohomology for physics in general was in the air lately; conferences entitled "Cohomological Physics" started to mushroom. In particular, in supergravity the importance of cohomology is clearly stated by several authors (e.g., [CDF]) but the description of the curvature tensor in nonholonomic case was never given until recently [Lp2], [GL3] and this is the whole point totally missed in attempts to describe Riemannian tensor on *Minkowski* superspaces as if they were holonomic (compare [Lp2], where computations are made under such assumptions with real SUGRA [GL3]) and such attempts stopped at $N = 1$, [RS].

Mechanics. In XIX century Hertz [H] divided the dynamical systems into *holonomic* (no constraints on velocities) and *nonholonomic* (with (nonintegrable) constraints on velocities). Analytical study of holonomic systems has progressed much further than that of nonholonomic. Moreover, only *linear* constraints are usually considered.

Though nonholonomic systems are important in various branches of applied physics end engineering, they have not been sufficiently studied in mathematics, even the best text books on analytical mechanics just mention several examples and pass to holonomic systems, cf. [A1], [A2]. One of the reasons: the qualitative study, such as stability problems, require

the notion of curvature tensor which is not defined for the nonholonomic systems in the literature, cf. [S], the best textbook for this purpose.

Examples of nonholonomic systems. 1) In **mechanics**. A ball on a rough plane; ballbearings and gyroscopes; any vehicle with wheels (the point at which the moving body is tangent to the surface has zero velocity). A car with a cruise control switched ON is an example of a nonholonomic system with a *nonlinear* constraint. For further (numerous) examples we refer the reader to Internet, see also [FK], [K], [E], [Gl] and references therein.

Related problems: general description of the movement and stability of a rapidly rotating hole body with a liquid or gyroscope inside it; orbital stability of a missile with liquid inside it. (The linear approximation does not work here.)

For several interesting practical applications (e.g., how a cat, when dropped, uses nonholonomicity to fall on its paws) see [M], [E].

In **electromechanics** (see, e.g., the book by Nejmark and Fufaev [NF] or [BF]) an early example was provided by Gaponov who, in 1952, found that a conductor connected to a gliding contact (as in a rotor collector) is equivalent to imposing nonholonomic constraints on the distribution of the electric current.

For comprehensive reviews of nonholonomic problems, see the books [NF] and [VG]. They list numerous problems which go back to Euler, Gauss, Carnot, Hertz, Appell, Caratheodory, Schouten, Synge, and many others. In [VG], and references therein (see also Agrachev's plenary talk at the 1994 International Congress of Mathematicians [Ag]), there are mentioned several problems of

2) **optimal control**, which lead to nonlinear constraints on generalized velocities (fields of cones, spheres, etc.).

3) the **magneto-hydrodynamics**: waves in plasma. Study of stability of such waves is extremely important for the development of thermonuclear power plants.

4) Recent studies of Komech, Spohn and Kunze (e.g., [KS]) show that electron represents a nonholonomic system with a (Coulomb, Maxwell, etc.) gauge as a nonholonomic constraint.

5) Numerous works on hypoelliptic differential equations (by Hörmander, Melin, Malliavin, Bismut, Bell; for a review see, e.g., [Ar]) also lead to nonholonomic distributions.

Related problems.

• Vershik was probably the first who noticed similarity of various distributions of solid bodies that appeared in optimal control (corresponding to the nonlinear nonholonomic constraints) with the "well-known" nonholonomic problems, cf. [CF], [VG] and [Ag].

• Supersymmetry appears even in seemingly non-super questions, like the study of the spectrum of the Schrödinger operator [L2] or relation between the Schrödinger operator and the KdV (Korteveg de Vries) operator [LX]. It does not always bring in nonholonomicity, but rather often, and usually unexpectedly.

• Recent studies of the Gelfand–Dickey bracket lead us to the discovery of a new class of simple Lie algebras of polynomial growth — generalization of Lie algebras of matrices of "complex size" [GL1], [GL2]. Simultaneously, together with Shchepochkina, we announced two classifications ([GLS], [LS]) of simple Lie superalgebras: (A) of vector fields and (B) stringy or superconformal superalgebras.

In November 1996 V. Kac wrote about problem (A) "*The problem of classification of all primitive* [close to simple, *P.G and DL*] *Lie superalgebras is a very interesting problem in itself (in view if recent progress in representation theory of finite dimensional simple Lie superalgebras, this is the last open problem from my Advances in Math. 1977 paper). However, my feeling is that it is too difficult to be solved in this millennium.*"

Solution of these problems [GLS], [LS] was considerably speedified by our computer-aided study.

• Recently V. Sergeev discovered a relation between nonholonomic manifolds that appear in thermodynamics with economics. He suggested mathematical models which give, among other things, a transparent explanation of such well known phenomena as economic growth in China and economic decline in post Soviet republics [SV].

• Related with Lie superalgebras of complex size generalizations of orthogonal polynomials (especially, hopefully, in several variables).

Amazingly, all the above topics are interrelated by common mathematics: Lie algebra cohomology and nonholonomic structures. M. Vasiliev with colleagues augmented our amazement: in the paper on super versions of Dunkl operators and Calogero–Sutherland model they showed that generalized Lie *super*algebras of supermatrices of complex supersize play the role of Poincaré algebras in N-extended SUGRAs for $N > 8$.

Accordingly, our research splits into several topics united by the usage of a common *MATHEMATICA*-based package SUPERLIE written by Grozman.

1. Study of nonholonomic mechanics: from supergravity to ballbearings. Applications to stochastic analysis, media with dislocations, and optimal control. Study of stability via cohomology.

2. Study of representation theory; in particular, classification simple Lie superalgebras of vector fields, stringy or superconformal, and generalizations of Lie algebras of matrices of complex size. Description of defining relations for them via cohomology. Applications to Lie algebras and superalgebras and their representations over fields of prime characteristic and over integers.

3. Application of representation theory to the study of integrability of differential equations (KdV, KP, Schrödinger, Liouville, Toda lattice, etc. and their superizations). Criteria of integrability expressed in terms of Lie algebra cohomology.

4. Classification of invariant differential operators.

In this talk we will briefly report two types of results: on SUGRA and on invariant operators.

§2. Formulation of problems

According to Arnold–Kozlov–Neishtadt [A2], the behavior of nonholonomic systems is often "surprising", and its quantitative study is handicapped by the lack of adequate tools. Let us recall some examples from [A2].

Consider a skate on the inclined plane. Where do you think it will move if pushed not directly downwards, but sideways? If we ignore friction it will never reach the floor but will oscillate between certain horizontal lines on the plane. Similarly, consider a ball rolling along the wall inside a vertical tube. It seems natural to expect the ball to descend on a spiral trajectory with increasing steepness. In reality, however, the ball will perform harmonic oscillations between two fixed vertical planes. Though individual solutions of nonholonomic systems are usually known, the stability questions are solved *ad hoc*.

Stability problems. The stability of a holonomic system can be studied in terms of the Riemann curvature tensor. The sign of the curvature indicates whether the geodesics converge or diverge: compare meridians on a sphere (positive curvature) with those on a trumpet = Lobachevsky plane (negative curvature).

For nonholonomic manifolds there was no such tool: in the literature the definition of the analogue of the Riemann tensor is only given in a few particular cases of little practical value, cf. [VG], [T]. For the traditional aanalytical approach see [RK].

Difficulties. In the last decades, dozens of publications on dynamical problems in nonholonomic mechanics appeared each year in technical journals, but very few in mathematical journals. The main reason: lack of adequate language needed to even formulate the problem of calculating the local curvature tensor (the analogue of the Riemann tensor) in intrinsic terms. The bulky coordinate expressions used by classics in 1920's, as well as now, hinder further progress. Vershik and Gerschkovich gloomily stated [VG] that there is no language to even formulate the integrability problems for nonholonomic systems with linear constraints (to say nothing about nonlinear constraints).

This is one of the reasons why this important field was abandoned by mathematicians, and even the interest and occasional works of such leaders as Faddeev, Griffiths, Chern and Arnold (see [VG], [Br], [A2]) did not attract any followers. Perhaps, only the latest works of Marsden [Ms] do not satisfy this pessimistic description.

§3. What is done: a notion, a formula, and a package

We know the importance of the notion of curvature in drawing very exact maps; application to stability were already mentioned. How to compute the curvature? In text books on geometry this is done in terms of Spencer cohomology. Same cohomology are used in Goldschmidt's criteria for (formal) integrability of differential equations.

Observe that these criteria [Br] are only given for "a half of" the cases. The point is that by a theorem of Cartan the symmetries of each differential equation are induced by either point transformations or contact transformations [KLV]. The latter case belongs to the realm of the simplest nonholonomic manifold — a contact one — and, apart from partial results of [T], nothing was known in this case.

A formula: Lie Algebra Cohomology. In lectures at a school at ICTP, Trieste, in 1990, Leites reformulated the definition of the Riemann tensor in terms of *Lie algebra cohomology* for any $\mathbb{Z}$-graded Lie algebra of finite depth (see §4). In these terms, the problems discussed above can be posed precisely and, in principle, solved. It becomes possible to generalize the description of the local curvature tensor for nonholonomic manifolds with any constraints, like the fields of surfaces.

Pilot Package. During 1992–94, Grozman developed a *MATHEMATICA*-based pilot package SUPERLIE for computing Lie algebras and their cohomology. The package embraces determination of various types of Lie algebras: matrix algebras, algebras of vector fields, algebras defined via generators and relations, via Cartan matrices, via Poisson brackets, etc.; the modules over them, and their homology and cohomology.

The mentioned above reformulation of Spencer cohomology in terms of Lie algebra cohomology means that much of the time-consuming calculations become considerably simplified and new ways to diminish the amount of calculations appear. However, the computation effort is still overwhelming: to compute the curvature tensor even for the simplest practical examples of, e.g., a ballbearing, or a bike, we need to develop the package further and complete its documentation to make available to engineers. On top of that *Mathematica* is irritatingly slow. Still, though it seems that an *ad hoc* written code for each particular task would be faster than our package, none managed so far. The advantage of our package is its universality and versatility. (For example, in [Gz], [LKW] bilinear differential operators acting in tensor fields invariant with respect to the general and volume preserving diffeomorphism groups are classified. Recently Grozman obtained new results in hamiltonian and contact cases, cf. §5.)

In spite of some drawbacks of the existing pilot package, we used it to correct several results found analytically by mathematicians (certain cohomology from [FL], [F]) and by physicists (certain Wess–Zumino constraint in supergravity [WB] turned out to be redundant). The last example is of particular interest.

The results, obtained completely or partly with the help of the package are described in [GL1], [GL2], [GKLP], [GLS] and used also in [LS], [LX]. Several more are in preparation.

Rival and not so rival teams. An alternative package (not so overwhelming but allegedly much faster) was being developed for several years in JINR, Dubna, by V. Kornyak, our coworker, in C. (So far his package can not do anything we can not.) For results of REDUCE-fans from Twente University see [LP], [PH], and refs. in [GKLP].

§4.0. The left hand sides of SUGRAs ([GL3])

Here we briefly explain how to solve the problem in deriving supergravity equations, SUGRA(N) — the analogues of Einstein's equations (EE) on an N-extended Minkowski superspace $\mathcal{M}(N)$ and what are our the criteria for distinguishing suitable Minkowski superspaces among other supermanifolds. Our considerations are in vacuum. As in twistor theory, we consider complex case, the physical reality to be recovered on a suitable real form of $\mathcal{M}(N)$. Our requirements: SUGRA(N) should be

(A) a differential equation of order ≤ 2 on the components;

(B) the component expansion of SUGRA(N) should contain the ordinary Einstein's equations.

For simplicity we assume that the supergroup of motions of $\mathcal{M}(N)$ is $\mathcal{G} = \mathcal{SL}(N|4)$ (though other possibilities can not be eliminated, cf. [GL2], [Ma]), so we wonder: what is the stationary subgroup $\mathcal{P}$ for which $\mathcal{M}(N) = \mathcal{SL}(N|4)/\mathcal{P}$?

As N grows, it becomes clear (by comparing the spectrum of cohomology for these supergroups of motion with the underlying Riemann tensor, as modules over $O(4)$) that to justify the above requirements (A) and (B) is only possible if we diminish $\mathcal{P}$, as GIKOS did, cf. [GI]. Then for $N > 3$ we see that the underlying manifold of $\mathcal{M}(N)$ is the direct product of several copies of the Minkowski space M (times, perhaps, an auxiliary space of a yet unclear merit); so SUGRA splits into the usual Einstein equations on each copy of M glued together by odd superfields. In particular, for $N = 4$ there are two copies of M.

For $N = 8$, in a most symmetric model, there are three copies of M, one of them distinguished (say, "our world"), the other two – perfectly interchangeable (in the model considered here; there are other possibilities) – mirroring, say, "heaven" and "hell"). These extra copies of the Universe (they MUST appear in our approach) embody an idea first, perhaps, voiced by A. Sakharov [Sa] on the necessity to consider of several copies of our Universe to describe physical reality. Another feature of nonholonomic nature of Minkowski *super*space is the preferred direction of time, an observation we derive by directly looking at the rattleback (Celtic stone); the universality of this observation for nonholonomic systems follows from recent elementary, and therefore wonderful, studies by A. Nordmark [N].

We start first with a presentation of Einstein's equations in a form convenient in what follows — as equations on conformally noninvariant components of the Riemann tensor, and represent the Riemann tensor as a section of the bundle whose fiber is certain *Lie algebra cohomology*. This is equivalent to the standard modern treatment of G-structures in differential geometry that uses *Spencer homology* [S] but can be generalized to treat nonholonomic structures, such as SUGRA. Then we generalize the notion of the G-structure to embrace *nonholonomic* manifolds, i.e., manifolds with nonintegrable distributions, see [H]. The structure of a nonholonomic manifold is encountered quite often. The simplest example

is the contact structure on a manifold, cf. [A1]. The applications range from the Cat's Problem to electro-mechanical devices [NF], [BF], [E], [CF], [K], [KS], [M]. For a moving account of nonholonomic problems and their history see [VG].

One can apply [S] to any supermanifold with a G-structure (such attempts are numerous in the literature) but the tensors obtained do not match the one physicists consider, cf. [WB]. We also offer a general method to derive constraints — analogues of Wess–Zumino constraints — for any N. Funnily, one of the conventional constraints for $N = 1$ is redundant: its cohomology class is zero. This demonstrates that a computer-aided study [GL5] is a must here: even when we know what to do and how, the amount of computations is too vast for a human. Even as skilled one as Poletaeva [P].

Though the notion of supermanifolds is past its 25-th birthday [L1], certain basics are, regrettably, insufficiently well-known yet. So we will recall them.

§4.1. Structure functions for nonholonomic structures

1.1. Nonholonomic (super)manifolds ([VG]). Let M be a manifold with a distribution D. Let

$$D = D_1 \subset D_2 \subset D_3 \subset \ldots \subset D_d \tag{1.0}$$

be the sequence of strict inclusions, where $D_i(x) = D_{i-1}(x) \cup [D_1(x), D_{i-1}(x)]$ for every $x \in M$ and d is the least number for which the sequence (1.0) stabilizes, i.e., such that $D_d(x) \cup [D_1(x), D_d(x)] = D_d(x)$. The number $d = d(M, D)$ is called *the nonholonomicity degree*. In case $D_d = TM$ the manifold M is called *completely nonholonomic*. Each pair: (M, D) with a nonintegrable D will be referred to as a *nonholonomic manifold* if $d \neq 1$. Here we will only consider completely nonholonomic (super)manifolds (the incomplete case corresponds to *nontransitive* generalized Cartan prolongs, cf. below, and the modifications needed to adjust general machinery to it are obvious).

Let $n_i(x) = \dim\ D_i(x)$. The distribution D is called *regular* if all the dimensions n_i are constant functions on M. In what follows we will only consider regular distributions.

With the tangent bundle over a nonholonomic manifold (M, D) we can naturally associate a sheaf of nilpotent Lie algebras as follows. At a point $x \in M$ set

$$\mathfrak{n}(x) = \bigoplus_{-d \leq i \leq -1} \mathfrak{n}_i(x), \quad \text{where } \mathfrak{n}_{-i}(x) = D_i(x)/D_{i-1}(x), \quad D_0 = 0. \tag{1.1}$$

Here d is the nonholonomicity degree of (M, D). Clearly, $\mathfrak{n}(x)$ is a well-defined Lie algebra.

1.1.1. A flat $(G, \mathfrak{n}(x))$-structure. This is a straightforward generalization of the notion of the flat G-structure (the latter implies, in particular, that every $\mathfrak{n}(x)$ is commutative, but not only that: the curvature tensor must also vanish). Namely, let $M = \mathbb{R}^n$ with a nilpotent $\mathbb{Z}$-graded Lie algebra structure, call it $\mathfrak{n}$. Let G be a subgroup of homogeneous (preserving the grading (1.1)) automorphisms of $\mathfrak{n}$. Let us identify the tangent space at a point $m \in M$ with M by means of a transformation from G. The preimages $\mathfrak{n}_{-1}(m)$ of $\mathfrak{n}_{-1}$ under this identification determine a distribution on M. This distribution together with the G-action on the accompanying flag $\mathfrak{n}(m)$ at each $m \in M$ will be called a *flat $(G, \mathfrak{n})$-structure*. Observe that this "flatness" and the obstructions to flatness introduced below differ drastically from their conventional counterparts.

In order to gauge the measure of this nonholonomic nonflatness, we need an analogue of Spencer cohomological complex, which, in turn, requires the following notion.

1.2. Generalized Cartan's prolongs. Given a $\mathbb{Z}$-graded nilpotent Lie algebra $\mathfrak{g}_- = \bigoplus_{0>i\geq -d} \mathfrak{g}_i$ and a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_-$ which preserves the $\mathbb{Z}$-grading of $\mathfrak{g}_-$, define the

i-th prolong of $(\mathfrak{g}_-, \mathfrak{g}_0)$ for $i > 0$ to be (here $S^\bullet = \oplus S^k$ and V^* is the dual of V):

$$\mathfrak{g}_i = [(S^\bullet(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap (S^\bullet(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-)]_i,$$

where the subscript in the right hand side singles out the component of degree i and the intersection is well-defined thanks to the fact that $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_- \subset \mathfrak{g}_-^* \otimes \mathfrak{g}_-$.

Define the *generalized Cartan's prolong* $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ as $\underset{i \geq -d}{\oplus} \mathfrak{g}_i$. By the routine arguments, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra. By the same arguments as for the G-structures, cf. [S], [Gz], [T], the space $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ is the space of obstructions to flatness of the nonholonomic supermanifold (M, D) and the elements of $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ will be called (as for the case $d = 1$) *structure functions*.

The space of structure functions naturally splits into homogeneous components whose degree is induced by the $\mathbb{Z}$-grading of $(\mathfrak{g}_-, \mathfrak{g}_0)_*$; generally, the minimal order of structure functions is $2 - d$. Let $C^s(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*) = \underset{k}{\oplus} C^{k,s}(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ be this splitting on the cochain level; the corresponding cohomology $H^{k,s}(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ are precisely the analogues of the Spencer cohomology and coincide with them for $\mathfrak{g}_- = \mathfrak{g}_{-1}$.

Example. Let $\mathfrak{g} = \mathfrak{csp}(2n), \mathfrak{g}_{-1} = R(\pi_1; 1), \mathfrak{g}_{-2} = R(0; 0)$; then $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n+1)$ and (here E is the exterior power operator)

$$C^{k,2}_{(\mathfrak{g}_-, \mathfrak{g}_0)_*} = \mathfrak{g}_{k-2} \otimes E^2(\mathfrak{g}_{-1}^*) \bigoplus \mathfrak{g}_{k-2-1} \otimes E^{2-1}(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-2}^*.$$

The number k here is the order of the structure functions. Unlike the holonomic case when order k structure functions are obstructions to identification of the k-th infinitesimal neighborhood of the point of M with that on the flat manifold, obstructions to flatness in nonholonomic case are not as lucidly interpreted.

1.3. Riemann-like and Weyl-like structures. Let $\hat{\mathfrak{g}}_0$ be the semisimple part of $\mathfrak{g}_0$ and $\hat{\mathfrak{g}}_* = (\mathfrak{g}_-; \hat{\mathfrak{g}}_0)_*$. The elements of $H^2(\mathfrak{g}_-; \hat{\mathfrak{g}}_*)$ are analogues of the *Riemann tensor* whereas the elements of $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ are analogues of the *Weyl tensor*, the structure functions of the conformal structure.

Theorem . (Weyl-like structures) *For* $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n+1)$ *all structure functions vanish.*

Remark . This is a conceptual reformulation of Darboux's theorem on the absence of parameters for the contact form.

1.4. Structure functions for projective structures. Let $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{-1}$ its standard (identity) representation. Then $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{vect}(n)$ and all structure function vanish ([LPS]); but if for coefficients of the above cohomology we take the subalgebra $\mathfrak{h} = \mathfrak{sl}(n+1) \subset \mathfrak{vect}(n)$ instead of $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, then the corresponding structure functions are nonzero and provide us with obstructions to integrability of what is called a *projective connection* ([G]).

More generally, for a $\mathbb{Z}$-graded subalgebra $\mathfrak{h} \subset (\mathfrak{g}_-, \mathfrak{g}_0)_*$, such that $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$ we will call $\underset{k}{\oplus} H^{k,2}(\mathfrak{g}_-; \mathfrak{h})$ *structure functions of the generalized projective structure of type* $\mathfrak{h}$. Here are examples:

Theorem . (in notations of [OV]; in $R(\varphi; k)$ the value on the center of $\mathfrak{g}_0$ stands after semicolon) 1) *Let* $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{vect}(n)$, $\mathfrak{h} = \mathfrak{sl}(n+1)$. *Then structure functions of order* 1 *and* 2 *vanish, structure functions of order* 3 *constitute the* $\mathfrak{g}_0$*-module* $R(\pi_1 + \pi_2 + \pi_{n-1}; 2)$.

2) *Let* $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n+1)$, $\mathfrak{h} = \mathfrak{sp}(2n+2)$. *Then the space of structure functions is* $R(\pi_1 + \pi_2; 3)$ *of order* 3.

1.5. Structure function for contact structures. Let $\omega_0 = \sum dp_i dq_i + \sum \varepsilon_j (d\xi_j)^2$ and $\omega_1 = \sum d\pi_i dq_i$ be the canonical even and odd, respectively, 2-forms (see [LPS]). Denote by $\mathfrak{k}(2n+1|m)$, resp. $\mathfrak{m}(n)$, the Lie superalgebra which preserve the contact structure given by the form $dt + \alpha$ and $d\tau + \beta$, where $d\alpha = \omega_0$, $d\beta = \omega_1$, respectively, and by $\mathfrak{sm}_\lambda(n)$ the deformation of $\mathfrak{sm}(n) = \mathfrak{svect}(n|n+1) \cap \mathfrak{m}(n)$ also described in [LPS].

Theorem . *For $\mathfrak{k}(2n+1|m)$, $\mathfrak{m}(n)$ and $\mathfrak{sm}_\lambda(n)$ all structure function vanish.*

§4.2. Structure functions of the N-extended Minkowski supermanifold

Recall that the ground field is that of complex numbers.

The G-structure of the Minkowski space can be viewed as either (pseudo) Riemannian or, equivalently, twistor structure. "Straightforward" superizations of these structures are distinct. They are considered in [M] (see also refs. therein) or [LPS] and [GL3], respectively. Neither of these superizations gives rise to what is accepted as supergravity. The reason is that a Minkowski superspace is still another superization of the Minkowski space and is naturally endowed with a nonholonomic structure.

2.1. What is an N-extended complexified Minkowski supermanifold. First of all, recall what was the "standard" model of complexified and a compactified Minkowski superspace $\mathcal{M}(N)$ before the GIKOS group (Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokachev) showed that one can not write SUGRA on it. Recall also that the "physical reasons" for the restrictions $N \leq 4$ for the Yang-Mills and $N \leq 8$ for the supergravity theories ([GI], [WB]) were put to doubt in [Va].

Recapitulations ([L2]). A *supermatrix* is a rectangular table with elements from a supercommutative superalgebra C with given sets of parities P_{row} and P_{col} of its rows and columns. The *size* of a matrix is $P_{row} \times P_{col}$. Usually, the parities are chosen so that the even rows and columns come first followed by the odd ones; such matrices are said to be of the *standard format*. For the square matrices we will only consider the cases when $P_{row} = P_{col}$ and will denote this set of parities by Par. The parity of the matrix unit — the matrix with an element $c \in C$ in the (i,j)-th position and 0's elsewhere — is defined to be $p(c) + P_{row}(i) + P_{col}(j)$. Hereafter in this paper $C = \mathbb{C}$.

Let $\mathfrak{gl}\,(Par)$ be the set of square matrices of size $Par \times Par$; let p be the number of 0's and q the number of 1's in Par. It is immediately clear, that for distinct Par's with the same p and q the Lie superalgebras $\mathfrak{gl}\,(Par)$ have *nonisomorphic* maximal nilpotent (say, upper triangular) subalgebras, though the algebras $\mathfrak{gl}\,(Par)$ themselves are isomorphic. Often, to consider the standard format only suffices, and $\mathfrak{gl}\,(Par)$ is abbreviated to $\mathfrak{gl}(p|q)$. In supergravity we MUST consider nonstandard formats as well.

The Lie superalgebra $\mathfrak{gl}(Par)$ is generated by the generators of its maximal nilpotent subalgebras, the upper and the lower triangular ones. Since $P_{row} = P_{col}$, the Lie subsuperalgebra of upper triangular matrices in $\mathfrak{gl}\,(Par)$is isomorphic to the Lie subsuperalgebra of lower triangular matrices and we will confine ourselves to one of them, denoted by $\mathfrak{n}$. The generators of $\mathfrak{n}$ are the elements just above (below) the main diagonal; we will denote the even generators of $\mathfrak{n}$ by white nodes and the odd generators by "grey" nodes, the nodes corresponding to commuting generators are disconnected, otherwise they are joined by a segment, cf. [GL2].

For instance, for the standard format, i.e. $\mathfrak{gl}(p|q)$, we have:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} . \quad \underbrace{0 - \cdots - 0}_{p-1 \text{ nodes}} - \otimes - \underbrace{0 - \cdots - 0}_{q-1 \text{ nodes}} \tag{2.0}$$

Consider the Lie supergroup $\mathcal{S}L(N|4)$ and its parabolic subsupergroup $\mathcal{P}$ corresponding to the two marked odd simple roots in the following system of simple roots (this means that $\mathcal{P}$ is generated by all the simple roots except for the marked negative ones):

$$0 - -- \overset{+}{\otimes} - - - \underbrace{0 - - \cdots - -0}_{N-1 \text{ nodes}} - - - \overset{+}{\otimes} - - -0 \tag{2.1}$$

Let $G = SL(N) \times SL_L(2) \times SL_R(2) \times \mathbb{C}^*$ be the Lie group corresponding to the 0-th term of the $\mathbb{Z}$-grading described by diagram (2.1), i.e., the degree of a marked root is equal to 1 the other simple roots being of degree 0; here indices L and R distinguish the "left" copy of $SL(2)$ from its "right" twin. In this case $\mathfrak{g}_0 = \mathfrak{g} = \mathfrak{o}(4) \oplus \mathfrak{gl}(N) = \mathfrak{sl}_L(2) \oplus \mathfrak{sl}_R(2) \oplus \mathfrak{gl}(N)$, $\mathfrak{g}_- = \bigoplus\limits_{-1 \geq i \geq -2} \mathfrak{g}_i$ with $\mathfrak{g}_{-1} = (\mathrm{id}_L \otimes Id) \oplus (\mathrm{id}_R \otimes Id^*)$, $\mathfrak{g}_{-2} = id_L \otimes id_R^*$, where id_j is the space of the standard (identity) representation of $\mathfrak{sl}_j(2)$; $j = L, R$ and Id is the space of the identity representation of $\mathfrak{sl}(N)$. The corresponding matrix representaion of $\mathfrak{p} = Lie(\mathcal{P})$ is of format $2|N|2$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix} \quad \text{with} \quad A_{11}, A_{33} \in \mathfrak{sl}(2); A_{22} \in \mathfrak{sl}(N) \tag{N_{st}}$$

(the odd elements are denoted by Q); it was first found for a particular real form of $\hat{\mathfrak{g}}_* = \mathfrak{g}_- \oplus \hat{\mathfrak{g}}_0$.

The *N-extended Minkowski superspace* $\mathcal{M}(N)$ is $\mathcal{S}L(N|4)/\mathcal{P}$ endowed with the natural $(G, \mathfrak{g}_-)$-structure. The conventional versions of the Minkowski superspace correspond to a certain real form of the (complex) superspace $\hat{\mathcal{M}}(N)$ with the $(\hat{G}, \mathfrak{g}_-)$-structure, i.e., the reduced $(G, \mathfrak{g}_-)$-structure. Clearly,

$$\hat{\mathcal{M}}(N) = \mathcal{P}/\hat{\mathcal{G}}, \text{ where } \hat{G} = SL(N) \times SL_L(2) \times SL_R(2).$$

GIKOS guessed (and we can prove) that these $\mathcal{M}(N)$ never satisfy our requirements (A) and (B) on SUGRA for $N > 1$. GIKOS considered an enlargement $\hat{\mathcal{R}}(N)$ of $\hat{\mathcal{M}}(N)$ defined as

$$\hat{\mathcal{R}}(N) = \mathcal{P}/\hat{\mathcal{G}}', \text{ where } \hat{G}' = Q \times SL_L(2) \times SL_R(2) \text{ and } Q \text{ is a parabolic subgroup of } SL(N).$$

In other words, from $\mathcal{P}$ we pass to a smaller parabolic subsupergroup, $\mathcal{P}'$, whose diagram has several middle roots marked as well. To satisfy (A) and (B), we have to test various $\mathcal{P}'$s. This is impossible without a computer.

We can prove that the diagram (2.0) regardless of the number of middle roots marked can not satisfy requirements (A) and (B). In particular, it is well known to physicists that even for $N = 1$ the standard model does not satisfy (B). So in order to satisfy requirements (A) and (B) we consider nonstandard formats.

For $N = 8$ the following two possibilities seem to be distinguished by their symmetry:

$$0- \overset{+}{\otimes} -0- \overset{+}{0} -0- \overset{+}{0} -0- \overset{+}{0} -0- \overset{+}{\otimes} -0 \quad Par = (001111111100) \tag{$8_{\text{"standard"}}$}$$

and

$$0- \overset{+}{0} -0- \overset{+}{\otimes} -0- \overset{+}{0} -0- \overset{+}{\otimes} -0- \overset{+}{0} -0 \quad Par = (000011110000) \tag{8_{triality}}$$

The "standard" model for $N > 0$ was originally of the form (N_{st}). To make presentation more graphic we will denote the odd matrix elements by Q.

Even though supergravitationists did use cumbersome notations and worked with bare hands, GIKOS guessed that in (N_{st}) for $N > 1$ there is no chance to get structure functions

satisfies (A) and (B) and in order to get EE-like equations, on has to enlarge the Minkowski superspace by taking for $N = 2$ a smaller A_{22}:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \text{ where } A_{22} = \begin{pmatrix} A_{22}^{11} & 0 \\ a_{22}^{21} & a_{22}^{22} \end{pmatrix}.$$

Elsewhere we will discuss the assumptions of Haag-Łopuszanski-Sohnius' theorem which lead to the Poincaré supergroup and its enlargement, $\mathcal{S}L(N|4)$. It could be that by, yet unmentioned, physical reasons some of the quotients of $\mathcal{S}L(N|4)$ have to be ruled out; so far these reasons are not formulated explicitly. At the moment we accept the correctness of the choice of the "twistor supergroup" $\mathcal{S}L(N|4)$ but investigate several possible quotients corresponding to various matrix realizations or, which is the same, to connected components of the flag supermanifolds. The stationary subgroups we will consider in this paper are the simplest ones: the Lie groups, subgroups of $O(4) \times SL(N)$.

Notice that even for the same stationary subgroup $O(4) \times SL(N)$ we can consider several realizations:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}.$$

The experience with the analogues of Einstein's equations on symmetric spaces [LPS] teaches us to consider the models of Minkowski superspace whose stationary subgroup is a product of several copies of $SL(2)$: otherwise the equations will not be of order 2.

With all these conventions we first have to study the following examples of Minkowski superspaces which we will denote more instructively by $\hat{\mathcal{M}}(Par)$ rather than $M(N)$:

For $N = 1$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where } A_{22} \text{ (resp. } A_{33}) = 0$$

and with $Par = (00100)$ and (00001), respectively.

For $N = 2$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Q_{21} & A_{22} & 0 & 0 \\ A_{31} & Q_{32} & A_{33} & 0 \\ Q_{41} & A_{42} & Q_{43} & A_{44} \end{pmatrix},$$

where $Par = (001100)$, (000011), and (100001), respectively. In the last case $A_{11} = A_{44} = 0$, otherwise $A_{ii} \in \mathfrak{sl}(2)$.

For $N = 4$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{22} = \begin{pmatrix} A_{22}^{11} & 0 \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix} \quad \text{and} \quad Par = (00111100), \qquad (N = 4_1)$$

(this corresponds to two copies of the Minkowski space united by a common superstructure) and

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Q_{21} & A_{22} & 0 & 0 \\ A_{31} & Q_{32} & A_{33} & 0 \\ Q_{41} & A_{42} & Q_{43} & A_{44} \end{pmatrix} \qquad (N = 4_2)$$

There is, naturally, the possibility with $Par = (00001111)$, as always.

For $N = 8$ the following cases seem to be the most interesting (symmetric) ones:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{ii} = \begin{pmatrix} A_{ii}^{11} & 0 \\ A_{ii}^{21} & A_{ii}^{22} \end{pmatrix}, \qquad (N = 8)$$

with $Par = (111100001111)$. This model corresponds to three copies of the Minkowski space appended by a 16-dimensional Gr_4^8, analogue of GIKOS' "harmonic" space and united by a common superstructure. Another possibility is

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{22} = \begin{pmatrix} A_{22}^{11} & 0 & 0 & 0 \\ A_{22}^{21} & A_{22}^{22} & 0 & 0 \\ A_{22}^{31} & A_{22}^{32} & A_{22}^{33} & 0 \\ A_{22}^{41} & A_{22}^{42} & A_{22}^{43} & A_{22}^{44} \end{pmatrix} \qquad (N = 8_1)$$

with $Par = (001111111100)$. Ever less symmetric possibilities are:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{33} = \begin{pmatrix} A_{33}^{11} & 0 & 0 & 0 \\ A_{33}^{21} & A_{33}^{22} & 0 & 0 \\ A_{33}^{31} & A_{33}^{32} & A_{33}^{33} & 0 \\ A_{33}^{41} & A_{33}^{42} & A_{33}^{43} & A_{33}^{44} \end{pmatrix} \qquad (N = 8_2)$$

with $Par = (000011111111)$ and

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Q_{21} & A_{22} & 0 & 0 \\ A_{31} & Q_{32} & A_{33} & 0 \\ Q_{41} & A_{42} & Q_{43} & A_{44} \end{pmatrix}, \quad \text{where} \quad A_{44} = \begin{pmatrix} A_{33}^{11} & 0 & 0 \\ A_{33}^{21} & A_{33}^{22} & 0 \\ A_{33}^{31} & A_{33}^{32} & A_{33}^{33} \end{pmatrix} \qquad (N = 8_3)$$

with $Par = (001100111111)$.

2.2. Theorem . *In Table (below) there are listed all the orders and weights of all the structure functions for the indicated $\hat{\mathcal{M}}(Par)$.*

The corresponding cocycles are listed in a detailed version. We numbered the cocycles so that the i-th cocycle of weight $w = (w_1, \ldots, w_k)$ and degree d is denoted by $\mathbf{R}_i^d(w_1, \ldots, w_k)$. We drop some of the indices if they are clear from the contents.

Clearly, there are more candidates on the role of $\hat{\mathcal{M}}(Par)$ for the N we have considered and for other N as well; the package allows the reader to perform corresponding calculations.

Remark . We could have stuck to the yet "standard" model with $\hat{G} = SL_L(2) \times SL(N) \times SL_R(2)$ which preserves $\varepsilon_L \otimes vol \otimes \varepsilon_R$, where ε_i for $i = L, R$ and vol are the volume elements preserved by $SL_i(2)$ and $SL(N)$, respectively. Then we could consider an equation similar to the equations (EE_0) from [LPS]; namely, consider a connection compatible with $\varepsilon_L \otimes vol \otimes \varepsilon_R$ and for a component R of weight 0 in the space of structure functions with respect to $\hat{G}$ take

$$\mathbf{R} = \lambda \varepsilon_L \otimes vol \otimes \varepsilon_R. \qquad (EE_0)$$

There might be several obstacles in writing (EE_0): such a tensor R might not exist, or, if it exists, the corresponding equation might be not of order 2, or the valency of R could be not the same as that of the rhs, the tensor preserved by G. The equations that appear in this way are considered in [LPS]; they correspond to higher spins of M. Vasiliev.

To interpret the supergravity in the same way as we have treated the Einstein Equations [LPS], define the supergravity equations as follows. On $\hat{\mathcal{M}}(N)$, the stationary subgroup (which coincides with $\hat{G}$) of the point preserves $\varepsilon_1 \otimes \varepsilon_2 \otimes \cdots \otimes \varepsilon_k$, where ε_i is the volume preserved by $SL_i(2)$, the i-th copy of $SL(2)$, in the 2-dimensional identity representation.

If there are several, say k, tensors of weight 0 we can take for $\mathbf{R}$ their linear combination and the coefficients of this combination determine a parameter which runs over $\mathbb{P}^{k-1}$.

Example: $N = 1$. The tensor $\mathbf{R}$ depends on a parameter, the ratio $a : b$ which runs over the projective line $\mathbb{P}^1$. Physicists call this parameter the *Gates–Sigel* parameter. On $\mathfrak{g}_{-1}$, there is the inner product given by the bracket. Notice that this product is even for $Par = (00100)$ and odd for $Par = (00001)$. The tacit choice was $Par = (00100)$. For $Par = (00100)$ the metric g on the Minkowski space M is the product of spinorial metrics ε_L and ε_R on the the maximal isotropic (with respect to the pairing) subspaces of $\mathfrak{g}_{-1}$, cf. [Ma].

The equation (EE_0) takes the form

$$a\mathbf{R}_1(00) + b\mathbf{R}_2(00) = \lambda g. \qquad (EE_0(a:b))$$

From the explicit form of the cocycles it is clear that the component expansion of the above equation does not contain (EE_0) for $Par = (00100)$, and only $\mathbf{R}_1(00)$ for $Par = (00001)$ has the right expansion.

Notice immediately, that for $(EE_0(a:b))$ to be well-defined we must demand that all structure functions of orders > 2 vanish. These conditions are the *Wess-Zumino constraints*. We have fewer of them than, say, in [WB]: one of the constraints is "harmless", its cohomology class is zero (like torsion of the Levi–Civita conneciton).

What shall we take for analogs of Ricci flatness? As for $N = 0$, these should be the vanishing conditions on the part of the Riemann tensor which does not belong to the conformal or Weyl tensor. For $N > 0$ there are several such components and we can equate to zero either or all of them. The equations are well-defined as integrability condition provided the constraints vanish.

If all structure functions of order 2 (and of lesser orders) vanish, the higher obstructions are well-defined and we can write an equation on them. For example, for $Par = (00100)$ we can equate to zero one or both of the equations:

$$\mathbf{R}(3,0) = 0, \quad \mathbf{R}(0,3) = 0.$$

Different choices correspond to different supergravities (minimal, flexible, etc.).

On how many parameters do these equation actually depend? The analysis will be also performed in the detailed version, let us only say here that the the above equations should be considered for a *symmetric connection compatible with the structure preserved by the stationary group of point* of $\hat{\mathcal{M}}(N)$, cf. [LPS]. There is only one such connection for all the $\hat{\mathcal{M}}(N)$. For $N = 0$ this suffices to single out the symmetric 10-dimensional part of the vierbein. For $N > 0$ we still have too many components after having singled out the symmetric part of the connection, but the requirement of vanishing of the structure function of orders lesser than that of EE (in our cases such are orders < 2) kills more components.

TABLE

$\deg(SF)\backslash Par$	(0000)	(00100)	(00001)
$\boxed{0}$	not defined	$(3,1),(1,3)$	—
1	—	$(1,0),(0,1)$	$(1,1)\boxed{1}^{+1}, (0,1)_1, \boxed{(0,3)_1}$
2	$(2,2),(0,0)$ $\boxed{(4,0),\ (0,4)}$	$(1,1),(0,0)^2$	$(0,0)^2,(2,2),$ $(1,0)_1, \boxed{(3,2)_1}$
3	not defined	$(3,0),(0,3)$	—

$\deg(SF)\backslash Par$	(001100)	(100001)	(000011)
$\boxed{-1}$	not defined	$(0,2),(2,0)$	not defined
$\boxed{0}$	$(222),(123),(321)$	$(0,1)_1,(1,0)_1$	$(222),(123)_1,(301)_1$
1	$(110)_1,(011)_1$	$\boxed{(0,0)},\boxed{(2,2)^2}$ $(1,1)^2,(1,0)_1^2,(0,1)_1^2$	$\boxed{(110)},\boxed{(031)_1}$ $(011)_1$
2	$(020)^2,\boxed{\boxed{(200)}}$ $(101),\boxed{\boxed{(002)}}$	$(2,2),(0,0)^3$ $(1,0)_1,(0,1)_1$	$(000),(220),(002)$ $\boxed{(200)},(020),(101)_1$
≥ 3	—	not defined	—

$\deg(SF)\backslash Par$	(00111100)	(00001111)
$\boxed{-1}$	$(1021),(1023),(1201),(2112),$ $(3201),(1122)_1,(2211)_1$	$(1122),(2211),(1001)_1$ $(1221)_1,(2112)_1,(3001)_1,(1003)_1$
0	—	—
1	$(0110),(0011)_1^{\boxed{2}+1},(1100)_1^{\boxed{2}+1}$	$(0011)^{\boxed{2}+1},(1100)^2,(0110)_1^{\boxed{2}+1}$
2	$(0000),(0220),(2000)$ $(1111),(0101)_1,(10010)_1,$ $(0002),(0020),(0200)$	$(0000)^2,(0022),(2200),$ $(0200),(0020)$ $(0101)_1,(1010)_1$
≥ 3	—	—

$\deg(SF)\backslash Par$	(111100001111)
$\boxed{-3}$	$(100001)^2,(100003),(300001),(100201),(102001)$ $(100221),(122001),(101112),(211101),(210012),(111111)$
$\boxed{-2}$	$(100122),(221001),(101211),(112101),(111012)$ $(210111),(030001),(100030)$
-1	—
0	—
1	$(000011)^4,(001100)^4,(110000)^4,(000110)_1^4,(011000)_1^4$
2	$(000000)^3,(000020),(000022)$ $(000200),(002200),(002000),(020000)$ $(220000),(000101)_1,(001010)_1,(010100)_1,(101000)_1$
≥ 3	—

Notations. The cocycles which also correspond to the "conformal" case – on shell – are $\boxed{\text{boxed}}$; the cocycles of small orders are all conformally invariant, such $\boxed{\text{orders}}$ are boxed; the cocycles *which only exist in the conformal case* are $\boxed{\boxed{\text{doubleboxed}}}$; the suscript 1 singles out odd cocycles; the exponent denotes the multiplicity of the cocycle; the multiplicity of conformally invariant vectors is boxed.

For structure functions of degree > 2 the entry "not defined" in the Table refers to the "Riemannian" case, i.e. to the $\hat{G}$-structure. It so happened that in these cases there are no conformal cohomologies which facilitates out plotting of the table. The dash — indicates that either there are no structure functions in this order or (if the degree is > 2 and the case is the Riemannian one) they are not defined.

Clearly, the Table is invariant under the change of parities $1 \leftrightarrow 0$ in Par.

§5. Bilinear differential operators invariant with repect to symplectomorphisms

5.1. Let ρ be a representation of the group $Sp(2m;\mathbb{R})$ in a V_ρ. A *tensor field of type* ρ on a $2m$-dimensional symplectic manifold M is an object t defined in each local coordinate system x, in which the symplectic form is of the canonical form $\omega = \sum_{i\leq m} dx_i \wedge dx_{2m+1-i}$, by the vector $t(x) \in V_\rho$ so that the passage to other coordinates, y (with the same property), is defined by the formula

$$t(y(x)) = \rho\left(\frac{\partial y(x)}{\partial x}\right) t(x).$$

Traditionally (see reviews [K1], [K2]) the fibers of the tensor bundles were considered finite dimensional, but Leites showed recently [LKW] that on supermanifolds it is natural and fruitful to consider infinite dimensional fibers: this leads to semi-infinite cohomology of supermanifolds. Similar problem for symplectic manifolds was not studied yet.

The space of smooth tensor fields of type ρ will be denoted by $T(\rho)$ or by $T(\lambda)$, where $\lambda = (\lambda_1, \dots, \lambda_m)$ is the lowest (or, for finite dimensional representations, highest, for convenience) weight of the irreducible representation ρ.

In what follows the letters ρ, σ, τ will denote irreducible representations of $Sp(2m;\mathbb{R})$ and letters λ, μ, ν their highest weights. (We should have concidered lowest weight only, but in this preliminary report we stick to finite dimensional representations.)

Examples of spaces of tensor fields: a) $T(0) = C^\infty(M)$;

b) $T(1,0,\dots,0) \cong \mathfrak{vect} \cong \Omega^1$ is the space of vector fields or (which is the same on any symplectic manifold thanks to the nondegeneracy of ω) the space of 1-forms on M;

c) $\prod^r = T(\underbrace{1,\dots,1}_{r-\text{many}},0,\dots,0)$ the space of primitive r-forms.

On the space $T_c(\rho)$ of tensor fields of type ρ with compact support, as indicated by the subscript, there is an invariant inner product

$$< \chi, \theta >= \int_M < \chi(x), \theta(x) > \omega_0^m, \qquad \text{(IP)}$$

where $< \cdot, \cdot >$ in the integrand is the $Sp(2m;\mathbb{R})$-invariant inner product on V_ρ.

A differential operator $B : T(\rho_1) \otimes T(\rho_2) \otimes \cdots \otimes T(\rho_n) \longrightarrow T(\tau)$ is called *n-ary* (*un*ary, *bi*nary, etc. for $n = 1, 2$, respectively). Such an operator B is called $Diff(M)$-*invariant* if it is uniquely expressed in all coordinate systems; it is $Diff_\omega(M)$-*invariant* if it is uniquely expressed in all coordinate systems in which the symplectic form is of the standard form.

5.2. The unary $Diff_\omega(M)$-invariant differential operators. All such operators are classifieded by A. N. Rudakov ([R1], [R2]):

0-th order: the multiplication by a number;

1-st order : the derivations of the primitive forms $d_+ : \prod^r \longrightarrow \prod^{r+1}$ and $d_- : \prod^{r+1} \longrightarrow \prod^r$ $(0 \leq r \leq m-1)$. These operators are compositions of the exterior differential $d : \Omega^p \longrightarrow \Omega^{p+1}$ and the projection onto the space of primitive forms; recall that $\Omega^p = \prod^p \oplus \prod^{p-2} \oplus \dots$ for $p \leq m$ and $\Omega^p \cong \Omega^{2m-p}$;

2-nd order: $d_2 = d_+ \circ d_- : \prod^r \longrightarrow \prod^r (1 \leq r \leq m)$.

Remark . Rudakov's theorem implies that other invariant operators that might spring to mind ($d_- \circ \omega d_-$, etc.) are multiples of the described ones.

5.3. The binary $Diff_\omega(M)$-invariant differential operators. Clearly, the operators $B^{*1} : T(\tau) \times T(\rho_2) \longrightarrow T(\rho_1)$ and $B^{*2} : T(\rho_1) \times T(\tau) \longrightarrow T(\rho_2)$ conjugate to (or 1- and 2-duals of) the invariant differential operator $B : T(\rho_1) \otimes T(\rho_2) \longrightarrow T(\tau)$ with respect to the inner product (IP) are also differential and invariant ones.

0-th order operators are obviously those of the form

$$Z(\chi, \theta) = pr(\chi(x) \otimes \theta(x)),$$

where $pr : V_\rho \otimes V_\sigma \longrightarrow V_\tau$ is the projection of the tensor product onto an irreducible component.

1-st order operators:

Theorem . *Any bilinear 1-st order (with respect to all arguments) $Diff_\omega(M)$-invariant differential operator $B : T(\lambda) \otimes T(\mu) \longrightarrow T(\nu)$ is a linear combination of the following cases* P1 – P8 *(some of which host several distinct operators) and the operators obtained from them by 1- and 2-dualizaton and transposition of the arguments.*

P1) $\lambda = (\underbrace{1, \dots, 1}_{p-\text{many } 1\text{'s}}, 0, \dots, 0)$; weights μ and ν differ by a unit in r places, $r \equiv p + 1$ mod 2. For $r \leq p + 1$ there exist operators of the form $Z(d_+\omega, \theta)$ and for $r \leq p - 1$ there exist operators of the form $Z(d_-\omega, \theta)$.

P2) The Lie derivative being restricted onto $Sp(2m; \mathbb{R})$-irreducible subspaces splits into several operators of the form $Z(d_\pm\omega, \theta)$ and an operator

$$L : Vect \times T(\rho) \longrightarrow T(\rho)$$

which cannot be reduced to operators of the form P1).

Remark . Observe that if $\xi \in \mathfrak{h}(M) \subset \mathfrak{vect}(M)$ is a Hamiltonian vector field, then, by identifying $\mathfrak{h}$ with $d\Omega^0$, we see that $d_+\xi = d_-\xi = 0$ and in this case L coincides with the Lie derivative. Therefore, L determines a representation of the Lie algebra $\mathfrak{h}(M)$ in the space $T(\rho)$. It is not difficult to show that the invariance of B is equivalent to its $\mathfrak{h}(M)$-invariance:

$$L(\xi, B(\chi, \theta)) = B(L(\xi, \chi), \theta) + B(\chi, L(\xi, \theta))$$

for any $\chi \in T(\rho), \theta \in T(\sigma), \xi \in \mathfrak{h}(M)$.

P3) $S^k\mathfrak{vect} \times S^l\mathfrak{vect} \longrightarrow S^{k+l-1}\mathfrak{vect}$ (clearly $S^k\mathfrak{vect} \cong T(k, 0, \dots, 0)$) is the Poisson bracket (a.k.a. the *symmetric Schouten's concomitant*) on (polynomial in momenta) functions on T^*M.

P4) λ, μ, ν are of the form $(2, 1, \dots, 1, 0, \dots, 0)$ each, with p, q and r non-zero numerical marks, respectively, such that $p+q+r \equiv 0 \mod 2$, $|p-q| \leq r \leq p+q$, and $p+q+r \leq 2m+2$.

If all inequalities are strict, then there exist **four distinct operators** defined on the spaces of such fields, otherwise there exist only **two** distinct operators. For $p + q + r \leq 2m$ two of these four or two operators are obtained as restrictions of the *Nijenhuis bracket*, or its conjugates, onto the subspaces

$$T(2, 1, \dots, 1, 0, \dots, 0) \subset \Omega^p \otimes_{C^\infty(M)} \mathfrak{vect}.$$

Remark . The remaining two operators (i.e., the ones which are not the restrictions of the Nijenhuis bracket) are new. We do not know anything about them except that they exist and the same applies to the following two cases P5) and P6).

P5) λ, μ are of the same form as for P4), $\nu = (3, 1, 1, \dots, 1, 0, \dots, 0)$. There exists one operator for $|p - q| + 1 \leq r \leq p + q - 1$, $\quad p + q + r \equiv 1 \mod 2$, $\quad p + q + r \leq 2m + 1$.

P6) λ, μ are the same as in 4), $\nu = (2, 2, 1, \dots, 1, 0, \dots, 0)$ with r non-zero entries. The operator exists under the same conditions on p, q, r as for P5).

P7) $\nu = (1, \dots, 1, 0, \dots, 0)$; whereas λ, μ and conditions on p, q, r are the same as in 5). In this case there exists a unique operator which is not reducible to operators of the form $d_{\pm}Z$. It is a restriction of the Nijenhuis bracket.

P8) $\lambda = (2, 0, \dots, 0)$; whereas μ and ν differ from each other by a unit at one place. There exists a unique such operator. Further on I'll give arguments which enable one to express it, in principle, explicitly.

5.4. 2nd order operators. Grozman could not *classify* such operators so far. However, he was lucky to find one new invariant operator, denoted in the literature Gz:

$$Gz : T(2, 0, \dots, 0) \times T(2, 0, \dots, 0) \longrightarrow T(2, 0, \dots, 0).$$

For $m = 1$ Grozman got the explicit expression for the operator Gz in 1976. Let me reproduce it. In coordinates x, y we have $\omega = dx \wedge dy$. Then

$$\begin{array}{l} Gz : a \cdot dx^2 + 2b \cdot dxdy + c \cdot dy^2, a' \cdot dx^2 + 2b' \cdot dxdy + c' \cdot dy^2 \mapsto \\ \frac{\partial^2 g}{\partial x^2} dx^2 + 2\frac{\partial^2 g}{\partial x \partial y} dxdy + \frac{\partial^2 g}{\partial y^2} dy^2 + (\{c, a'\} - \{a, c'\}) dxdy + \\ \left(\frac{\partial^2 a}{\partial y^2} - 2\frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 c}{\partial x^2}\right)(a' dx^2 + 2b dxdy + c' dy^2) + (\{a, b'\} - \{b, a'\}) dx^2 + \\ \left(\frac{\partial^2 a}{\partial y^2} - 2\frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 c}{\partial x^2}\right)(a dx^2 + 2b dxdy + c dy^2)(\{b, c'\} - \{c, b'\}) dy^2, \end{array}$$

where $g = ac' - 2bb' + ca'$ and $\{\cdot, \cdot\}$ is the Poisson bracket. An explicit form of Gz for $m > 1$ is to be found.

5.5. Sketch of the proof of Theorem. Set $y_i = x_{2m+1-i} (1 \leq i \leq m), \partial_i = \frac{\partial}{\partial x_i}, \delta_i = \frac{\partial}{\partial y_i}$. Denote the elements of the Lie algebra $\mathfrak{sp}(2m; \mathbb{R}) \subset \mathfrak{h}(M)$ by

$$e^{ii} = y_i \partial_i, \quad e_{ii} = x_i \delta_i, \qquad e^i_j = x_j \partial_i + y_i \delta_j.$$

Then, clearly,

$$e^{ij} = e^{ji} = y_i \partial_j + y_j \partial_i \text{ and } \quad e_{ij} = e_{ji} = x_i \delta_j + x_j \delta_i \text{ for } i \neq j.$$

Let $I(\rho)$ be the space of differential operators from $T(\rho)$ into $C^\infty(M)$ with constant coefficients, i.e.,

$$I(\rho) = \{\sum P_i(\partial, \delta) u_i \mid u_i \in V_\rho^* \cong V_\rho\}.$$

The grading in $I(\rho)$ is induced by that in the space of polynomials P_i's, i.e., $I(\rho)_0 \cong V_\rho$. Define the pairing $I(\rho) \times T(\rho) \longrightarrow \mathbb{R}$ by the formula

$$< Pu, x >= P(< u, \chi(x) >)|_{x=0}.$$

On $I(\rho)$, define the $\mathfrak{h}(M)$-action, dual to the action on $T(\rho)$, via L. Now, to describe the invariant operators it suffices to find all the $\mathfrak{h}(M)$-morphisms $I(\tau) \longrightarrow I(\rho) \otimes_{\mathbb{R}} I(\sigma)$. It turns out that such a morphism is completely defined by the image of the highest vector $v \in V_\tau = I(\tau)_0$. Here we have fixed a Borel subalgebra $\{\sum_{i \leq j} a_{ij} x_j \partial_i\} \bigcap \mathfrak{sp}(2m; \mathbb{R})$ so that $w \in I(\rho) \otimes_{\mathbb{R}} I(\sigma)$ can be the image of a highest weight singular vector if and only if

$$e^i_{i+1} w = 0 \text{ for } 1 \leq i \leq m-1 \text{ and } e^{m,m} w = 0 \qquad \text{(conditions on } w \text{ to be a highest vector)}$$

and

$$(x_1^2 \delta_1) w = 0. \qquad \text{(conditions of } \textit{singularity} \text{ of the vector)}$$

The degree of $w \in I(\rho) \otimes_{\mathbb{R}} I(\sigma)$ is equal to the order of the corresponding differential operator. The general form of a vector of degree 1 is

$$w = \sum_{i \leq m} \partial_i' z_i^0 + \delta_i' t_i^0 + \partial_i'' z_i^1 + \delta_i'' t_i^1,$$

where $z_i^j, t_i^j \in V_\rho \otimes V_\sigma$, $\partial'(u \otimes v) = \partial u \otimes v$, $\partial''(u \otimes v) = u \otimes \partial v$. If w is a highest vector, then all vectors z, t are expressed in terms of z_1^0, z_1^1 which should satisfy

$$e_{i+1}^i z_1^j = 0 \text{ for } 2 \leq i \leq m-1, (e_2^1)^2 z_1^j = 0, e^{m,m} z_1^j = 0.$$

The condition $(x_1^2 \delta_1) w = 0$ is equivalent to the equation

$$e_{1,1}' z_1^0 + e_{1,1}'' z_1^1 = 0,$$

where (double) prime means that the operator acts only on the first (second) multiple of the tensor product.

Grozman succeeded to define all the cases, where the above system possesses a solution in $V_\rho \otimes V_\sigma$; though in certain cases he was not able to find the solution itself.

Here is an example of a sucessfully solved case (case 8)):

$$\lambda = (2, 0, \dots, 0), \quad \nu = (\mu_1, \dots, \mu_{k-1}, \mu_k + 1, \mu_{k+1}, \dots);$$

the case $\nu_k = \mu_k - 1$ is dual to this one. Let $u_0 \in V_\rho$ be a highest vector, then

$$u_0 \otimes v - \frac{1}{2} \sum_{2 \leq i \leq k} e_1^i u_0 \otimes e_i^1 v \in V_\rho \otimes V_\sigma$$

is a highest vector of weight $(\nu_1 + 1, \nu_2, \dots, \nu_m)$. We conclude that

$z_1^0 = u_0 \oplus e_{11} v - \sum_{2 \leq i \leq k} e_1^i u_0 \otimes e_{11} e_i^1 v - \frac{1}{2} \sum_{2 \leq i < j \leq m} e_1^i e_1^j u_0 \otimes (e_{ij} + e_{1j} e_1^i + e_{1i} e_j^1) v -$
$\frac{1}{2} \sum_{2 \leq i \leq k} (e_1^i)^2 u_0 \otimes (e_{ii} + e_{1i} e_i^1) v + \frac{1}{2} \sum_{2 \leq i \neq j \leq m} e_1^i e_{1j} u_0 \otimes (e_i^j + e_1^j e_i^1) v + (\nu_i + e_1^i e_i^1) v +$
$\frac{1}{4} \sum_{2 \leq i \leq m} e_{ii} (e_1^i)^2 u_0 \otimes + \frac{\nu_1 - 1}{2} \sum_{2 \leq i \leq k} e_{11} e_1^i u_0 \otimes e_i^1 v,$

and

$$z_1^1 = -e_{11} u_0 \otimes v + \sum_{2 \leq i \leq k} e_{11} e_1^i u_0 \otimes e_i^1 v.$$

5.6. Conjectures. 1) *The operator Gz is a particular case of a more general operator :*

$$Gz_{r,s} : T(2, \underbrace{1, \dots, 1}_{r-\text{many}}, 0, \dots, 0) \times T(2, \underbrace{1, \dots, 1}_{s-\text{many}}, 0, \dots, 0) \longrightarrow T(2, \underbrace{1, \dots, 1}_{r+s-\text{many}}, 0, \dots, 0).$$

2) *Operators of order > 2 are compositions of operators of orders ≤ 2. There are no operators of order > 5.*

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